Note on the sub-Slutsky matrix and optimal commodity taxation analysis

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Abstract

This paper analyzes the property of the sub-Slutsky matrix used in the analysis of the standard optimal commodity taxation model. It shows that the rank of the sub-Slutsky matrix is $N-1$, where N is the number of goods. This result implies that the marginal welfare cost of non-lump-sum taxation is strictly positive.

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1 Introduction

Consider the standard optimal commodity taxation model. Assume that there are N goods and let goods 1 be the untaxed goods and $t \in R^{N-1}$ be the optimal commodity tax vector. Let $x \in R^N$ be a goods vector and $\widetilde{S}$ be the $(N-1) \times (N-1)$ matrix formed by deleting the first column vector and the first row vector of the Slutsky matrix S. Then, at the optimal commodity tax vector t, we have the following relationship:

$$\widetilde{S}t = -\gamma x \tag{1}$$

where γ is a measure of the marginal welfare cost of distortionary taxation. Eq. (1) is the famous Samuelson equal proportional reduction rule (Atkinson and Stiglitz 1980). One natural question is whether this sub-matrix $\widetilde{S}$ is invertible or not. Some textbooks state that it is assumed that $\widetilde{S}$ is invertible (Myles 1995). The authoritative review paper on optimal taxation theory states that the determinant of $\widetilde{S}$ is not generally zero, but it does not provide a proof (Auerbach and Hines 2002). Another textbook applies Cramer's rule to Eq. (1), which implicitly assumes that $\widetilde{S}$ is non-singular (Jha 1998). It might suppose that if we need to assume that the sub-Slutsky matrix is non-singular, we might also need to impose some restriction on preference, and, thus, the utility function.

The non-singularity of the sub-Slutsky matrix $\widetilde{S}$ is also related to the value of γ, the marginal welfare cost of non-lump-sum taxation. From the negative semi-definiteness of the Slutsky matrix, it is straightforward to show that γ is not strictly negative. However, since the Slutsky matrix can never be negative definite, it is not obvious whether γ cannot be zero or not. Thus, the question arises as to whether, under the standard assumption on the preference (strictly quasi-concave utility function), the marginal welfare cost of non-lump-sum tax can be zero. Obviously, if $\widetilde{S}$ is invertible, then we can prove that γ is not zero.

The purpose of this note is to show that the sub-Slutsky matrix is non-singular and the marginal cost of non-lump-sum taxation is always strictly positive under the standard assumption on the utility function (strictly quasi-concave utility function, twice continuously differentiable utility function and differential demand function).

2 Analysis

2.1 Set-up

Consider an economy in which there are N goods. Let $q \in R_+^N$ be the consumer price vector and m be income. Let $x \in R_+^N$ be the goods vector and let $u(x) : R_+^N \to R$ be the utility function. In this paper, the following assumptions are made on the utility function:

(A1) The utility function is strictly quasi-concave and C^2.

(A2) For any positive goods vector $x \in R_+^N$, the partial derivative of the utility function $\partial u(x)/\partial x_i$ is strictly positive.

(A3) The Marshallian demand function is differentiable with respect to all prices and income.

(A4) For any strictly positive consumer price vector, $q \in R_{++}^N$, the demand is always strictly positive for all goods.

Assumption (A1) is standard and the assumption of C^2 is necessary since we take derivatives of the first-order conditions. (A2) is a non-satiation assumption. It implies that the prices are always strictly greater than zero. (A3) is necessary to take a derivative of the demand function and to form the Slutsky matrix. (A4) is used to avoid considering the corner solution.

To simplify the notation, let I_k be the identity matrix with dimension k and $\bar{0}_k$ be the zero column vector with dimension k. The $(N+1)\times(N+1)$ matrix H is also defined as follows:

$$H = \begin{pmatrix} u_{xx} & -q \\ -q' & 0 \end{pmatrix}$$

where u_{xx} is the $N\times N$ Hessian matrix of the utility function, q is the price vector and $q' = [q_1, q_2, \dots, q_N]$. The matrix H is the bordered Hessian matrix of the first-order conditions of the utility maximization. Now we prove the following observation.

Observation 1 *The Marshallian demand function and the Lagrangian multiplier of the first-order utility maximization are differentiable with respect to* (p, m) *if and only if* H *is not singular.*

Proof For proof of the sufficient part, note that the first-order conditions of utility maximization are:

$$\partial u/\partial x_i = \lambda q_i \text{ for } i = 1, 2, \dots, N$$
$$m - \sum_{i=1}^{N} q_i x_i = 0$$

where λ is the Lagrangian multiplier of the budget constraint. Let the solution of the above first-order condition be (x, λ). From the implicit function theorem, if the determinant of H is non-zero, x and λ become differentiable with respect to q and m. Thus, we can prove the sufficient condition part. For the necessary condition part, suppose that x and λ are differentiable with respect to q and m. Then, substituting x and λ into the first-order condition, and taking the derivative with respect to q and m, we have:

$$H\begin{pmatrix} \frac{\partial x_1}{\partial q_1} & \cdots & \frac{\partial x_1}{\partial q_N} & \frac{\partial x_1}{\partial m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_N}{\partial q_1} & \cdots & \frac{\partial x_N^*}{\partial q_N} & \frac{\partial x_N}{\partial m} \\ \frac{\partial \lambda}{\partial q_1} & \cdots & \frac{\partial \lambda^*}{\partial q_N} & \frac{\partial \lambda}{\partial m} \end{pmatrix} = \begin{pmatrix} \lambda I_N & \overline{0}_N \\ X' & -1 \end{pmatrix} \equiv Q \tag{2}$$

where $X' = [x_1, x_2, ..., x_N]$

First, it is straightforward to show that the rank of Q is $N+1$. Suppose that it is not. Then, there is a non-zero vector $a' = [a_1, a_2, \ldots, a_{N+1}]$ that satisfies $Qa = 0$. This means that:

$$\lambda a_i = 0 \text{ for } i = 1, 2, \ldots, N$$

$$\sum_{i=1}^{N} x_i a_i - a_{N+1} = 0$$

Because of Assumption (A2), $\lambda > 0$. This implies that $a_i = 0$ for all $i = 1, 2, \ldots, N+1$. This is the contraction. Thus, the rank of Q must be $N+1$ and $N+1$ columns of Q are linearly independent.

Note that Eq. (2) shows the $N+1$ column vectors of H are transformed to the linearly independent $N+1$ column vectors of Q. This implies that the $N+1$ column vectors of H must be linearly independent. Thus, the matrix H is non-singular.■

The above observation and Assumption (A3) guarantee that we can consider the inverse matrix of H. Using this property, we prove that the sub-Slutsky matrix is non-singular.

2.2 Property of the sub-Slutsky matrix

In this section, we examine the property of the Slutsky sub-matrix. To derive the property of the sub-matrix of the Slutsky matrix, we use the following observation.

Observation 2 *The rank of the Slutsky matrix is* $N-1$

Proof See Appendix A.

For notation, let the i-th column of the full Slutsky matrix be $v_i \in R^N$. In addition, let $\widetilde{v}_i \in R^{N-1}$ be the column vector that is obtained by deleting the first element of v_i. Let S be the Slutsky matrix. Note that $\widetilde{S}$ is the $(N-1) \times (N-1)$ matrix formed by eliminating the first column vector and the first row vector of S. Let $\widetilde{\widetilde{S}}$ be the $N \times (N-1)$ matrix formed by eliminating the first column vector of S. Let q be the price vector and let $\widetilde{q} \in R^{N-1}$ be the price vector obtained by deleting the first element of q. Let p be the producer price vector and t_i the commodity tax on goods i $(i = 2, 3, \ldots, N)$. Then, $q' = [p_1, p_2 + t_2, p_3 + t_3, \ldots, p_N + t_N]$

Since the compensated demand functions are homogeneous of degree zero with respect to prices, we have:

$$Sq = \overline{0}_N$$

Since $q_1 = p_1 > 0$, this implies that v_1 is linearly dependent on $[v_2, v_3, \ldots, v_N]$. Our strategy for the proof is that we prove that if the sub-matrix $\widetilde{S}$ is singular, then $v_2, v_3, \ldots, v_N$ are also linearly dependent. This means that the rank of the Slutsky matrix is strictly less than $N-1$. Thus, we have a contradiction.

Note that from the relation $Sq = \overline{0}_N$, we have $-q_1\widetilde{v}_1 = \widetilde{S}\widetilde{q}$. This implies that $\widetilde{v}_1 = -(1/q_1)\widetilde{S}\widetilde{q}$. Note that $\widetilde{v}_1$ consists of the 2nd to the N-th rows of the first column vector of S. Since S is symmetric, this also means that $\widetilde{v}_1'$ represents the 2nd to the N-th columns of the first row vector of S.

Now suppose that the sub-matrix $\widetilde{S}$ is singular. Then, there is non-zero vector $\alpha \in R^{N-1}$ that satisfies the following equation:

$$\widetilde{S}\alpha = \overline{0}_{N-1}$$

$$\text{where } \alpha \neq \overline{0}_{N-1}$$

Since $\widetilde{v}_1'$ is the 2nd to the N-th columns of the first row of S, if we can show that $\widetilde{v}_1'\alpha = 0$, then $\widetilde{\widetilde{S}}\alpha = \overline{0}_N$. In this case, this implies that the last $N-1$ columns of the matrix S are also linearly dependent. Now calculate $\widetilde{v}_1'\alpha$. Since $\widetilde{S}$ is symmetric, $\widetilde{v}_1' = -(1/q_1)\widetilde{q}'\widetilde{S}$. Thus,

$$\widetilde{v}_1'\alpha = -(1/q_1)\widetilde{q}'\widetilde{S}\alpha$$

$$= 0$$

Therefore, $\widetilde{\widetilde{S}}\alpha = \overline{0}_N$. This implies that the last $N-1$ columns of S are linearly dependent. However, this implies that the rank of S is strictly less than $N-1$. On the other hand, the rank of S should be $N-1$. Thus, we have a contradiction. Therefore, the sub-matrix $\widetilde{S}$ must be non-singular. Clearly, we can apply our argument to any sub-Slutsky matrix that is formed by deleting the i-th column vector and i-th row vector of the full Slutsky matrix.

Proposition 1 *When Assumptions (A1), (A2), (A3) and (A4) are satisfied, then the sub-Slutsky matrix formed by deleting the i-th column vector and i-th row vector of the full Slutsky matrix is non-singular.*

Once we establish that $\widetilde{S}$ is invertible, then it is also straightforward to show that the Lagrangian multiplier of the government budget constraint is strictly positive, as long as positive government revenue is needed. First, from the negative semi-definiteness of the Slutsky matrix, γ is positive or equal to zero. However, if γ is equal to zero, then, since $\widetilde{S}$ is invertible, this implies that all t_i are equal to zero. However, such taxes do not satisfy the government budget constraint. Thus, we have the following proposition.

Proposition 2 *Under Assumptions (A1), (A2), (A3) and (A4) and a positive required tax revenue, the marginal welfare cost of non-lump-sum taxation is strictly positive.*

3 Conclusion

This note demonstrates the property of the sub-Slutsky matrix that researchers encounter in the standard optimal commodity taxation model. It shows that under the standard assumptions on the demand function and the utility function, the sub-Slutsky matrix is non-singular.

Appendix A

This appendix proves that the rank of the Slutsky matrix is $N-1$. First, let q be the price vector. Then, the first-order conditions are:

$$U_x - \lambda q = \overline{0}_N$$

$$m - qx = 0$$

By taking the derivative of the above first-order conditions with respect to q, we have:

$$\begin{pmatrix} \frac{\partial x_1}{\partial q_1} & \dots & \dots & \frac{\partial x_1}{\partial q_N} \\ \dots & \dots & \dots & \dots \\ \frac{\partial x_N}{\partial q_1} & \dots & \dots & \frac{\partial x_N}{\partial q_N} \\ \frac{\partial \lambda}{\partial q_1} & \dots & \dots & \frac{\partial \lambda}{\partial q_N} \end{pmatrix} = H^{-1} \begin{pmatrix} \lambda I_N \\ X' \end{pmatrix}$$

where $X' = [x_1, x_2, \dots, x_N$.

For the income effect, we have:

$$\begin{pmatrix} \frac{\partial x_1}{\partial m} \\ \dots \\ \frac{\partial x_N}{\partial m} \\ \frac{\partial \lambda}{\partial m} \end{pmatrix} = H^{-1} \begin{pmatrix} \overline{0}_N \\ -1 \end{pmatrix}$$

Thus,

$$\begin{pmatrix} \frac{\partial x_1}{\partial q_1} & \dots & \dots & \frac{\partial x_1}{\partial q_N} \\ \dots & \dots & \dots & \dots \\ \frac{\partial x_N}{\partial q_1} & \dots & \dots & \frac{\partial x_N}{\partial q_N} \\ \frac{\partial \lambda}{\partial q_1} & \dots & \dots & \frac{\partial \lambda}{\partial q_N} \end{pmatrix} + \begin{pmatrix} x_1 \frac{\partial x_1}{\partial m} & x_2 \frac{\partial x_1}{\partial m} & \dots & x_N \frac{\partial x_1}{\partial m} \\ \dots & \dots & \dots & \dots \\ x_1 \frac{\partial x_N}{\partial m} & x_2 \frac{\partial x_N}{\partial m} & \dots & x_N \frac{\partial x_N}{\partial m} \\ x_1 \frac{\partial \lambda}{\partial m} & x_2 \frac{\partial \lambda}{\partial m} & \dots & x_N \frac{\partial \lambda}{\partial m} \end{pmatrix} = H^{-1} \begin{pmatrix} \lambda I_N \\ 0 \end{pmatrix}$$

Using the structure of a bordered Hessian matrix, without loss of generality, we can write the inverse of H as:

$$H^{-1} = \begin{pmatrix} A & B \\ B' & c \end{pmatrix}$$

where A is $N \times N$, B is $N \times 1$, and c is 1×1. Thus, the Slutsky matrix is:

$$\begin{aligned} \begin{pmatrix} \frac{\partial x_1}{\partial q_1} & & \frac{\partial x_1}{\partial q_N} \\ & & \\ \frac{\partial x_N}{\partial q_1} & & \frac{\partial x_N}{\partial q_N} \end{pmatrix} + \begin{pmatrix} x_1 \frac{\partial x_1}{\partial m} & x_2 \frac{\partial x_1}{\partial m} & \dots & x_N \frac{\partial x_1}{\partial m} \\ \dots & \dots & \dots & \dots \\ x_1 \frac{\partial x_N}{\partial m} & x_2 \frac{\partial x_N}{\partial m} & \dots & x_N \frac{\partial x_N}{\partial m} \end{pmatrix} &= \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} \lambda I_N \\ 0 \end{pmatrix} \\ &= \lambda A \end{aligned}$$

Thus, the Slutsky matrix is λA.

Note that since the matrix H is symmetric, the inverse of H is also symmetric. Thus, A is symmetric. Next, from the definition of H^{-1}, we have:

$$\begin{pmatrix} U_{xx} & -q \\ -q' & 0 \end{pmatrix} \begin{pmatrix} A & B \\ B' & c \end{pmatrix} = I_{N+1}$$

This implies:

$$q'A = \overline{0}'_N \tag{3}$$

$$U_{xx}A = qB' + I_N \tag{4}$$

¿From Eq. (3) and the symmetry of A, $Aq = \overline{0}_N$. This implies that the column vectors of A are linearly dependent and the maximum rank of A is $N-1$. Next, we show that the rank of A is $N-1$. To show the rank of A, we investigate the rank of $qB' + I_N$, the right-hand side of Eq. (4). We denote $qB' + I_N$ as D. Since the maximum rank of A is $N-1$, the rank of D is also at most $N-1$. This implies that the column vectors of $qB' + I_N \equiv D$ are linearly dependent. Now consider $N-1$ column vectors of D that can be formed by deleting any one column vector of D. We prove the following property.

Corollary 1 *Let $\widetilde{D}$ be the matrix with dimensions $N \times N-1$ formed by eliminating any one column of D. Then, the rank of $\widetilde{D}$ is $N-1$.*

Suppose that it is not. Then this implies that the column vector of $\widetilde{D}$ is linearly dependent. Suppose that the matrix $\widetilde{D}$ is formed by deleting the first column of D. This implies that there is a non-zero vector $\alpha' = [\alpha_2, \alpha_3, \ldots, \alpha_N]$ that satisfies $\widetilde{D}\alpha = 0_N$. By the definition of $\widetilde{D}$, this implies that:

$$\begin{pmatrix} 0 \\ \alpha_2 \\ \ldots \\ \ldots \\ \alpha_N \end{pmatrix} = \begin{pmatrix} q_1 \sum_{i=2}^N \alpha_i b_i \\ q_2 \sum_{i=2}^N \alpha_i b_i \\ \ldots \\ \ldots \\ q_N \sum_{i=2}^N \alpha_i b_i \end{pmatrix}$$

Since q_1 is not equal to zero, $\sum_{i=2}^N \alpha_i b_i$ should be equal to zero. This implies that $\alpha' = 0_{N-1}$. This contradicts the initial assumption that the $N-1$ column vector is linearly dependent. Note that our argument holds for any $\widetilde{D}$ that is formed by deleting any one column vector of D. Therefore, the rank of $\widetilde{D}$ is $N-1$. ■

Now we prove the following property.

Corollary 2 *The rank of the Slutsky matrix is $N-1$.*

Since the rank of $\widetilde{D}$ is $N-1$, the rank of D is at least $N-1$. Since D cannot be of full rank, the rank of D must be $N-1$. On the other hand, from Eq. (4) and using the symmetry of A, we can write that

$$AU_{xx} = Bq' + I_N$$

This implies that the N-th column vector is projected to the matrix with rank $N-1$. This means that the rank of A must be at least as great as $N-1$. On the other hand, since the matrix A cannot be of full rank, the rank of A must be $N-1$.

Finally, since the Slutsky matrix is λA, the rank of the Slutsky matrix is $N-1$. ■

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