Chapter 7: Contents

- the closed economy Solow model
- how a country's standard of living depends on its saving and population growth rates
- how to use the "Golden Rule" to find the optimal saving rate and capital stock

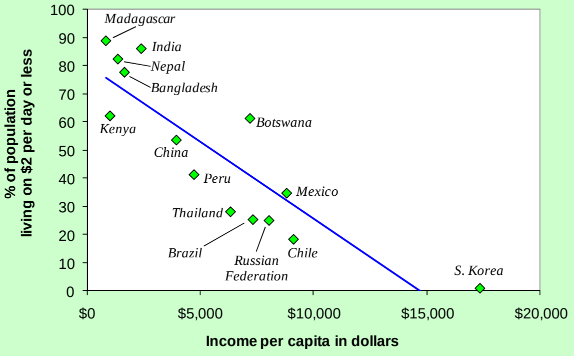
Why growth matters

- Data on infant mortality rates:
  - 20% in the poorest 1/5 of all countries
  - 0.4% in the richest 1/5
- In Pakistan, 85% of people live on less than $2/day.
- One-fourth of the poorest countries have had famines during the past 3 decades.
- Poverty is associated with oppression of women and minorities.

Economic growth raises living standards and reduces poverty….

Income and poverty in the world

selected countries, 2000



Why growth matters

- Graph of life expectancy and GDP per capita
- Graph of infant mortality and GDP per capita
- Dynamic movement of health and income

Why growth matters

- Anything that effects the long-run rate of economic growth – even by a tiny amount – will have huge effects on living standards in the long run.

annual growth rate of income per capita	percentage increase in standard of living after…		
	…25 years	…50 years	…100 years
2.0%	64.0%	169.2%	624.5%
2.5%	85.4%	243.7%	1,081.4%

Why growth matters

- If the annual growth rate of U.S. real GDP per capita had been just one-tenth of one percent higher during the 1990s, the U.S. would have generated an additional $496 billion of income during that decade.

The Solow model

- due to Robert Solow, won Nobel Prize for contributions to the study of economic growth
- a major paradigm:
  - widely used in policy making
  - benchmark against which most recent growth theories are compared
- looks at the determinants of economic growth and the standard of living in the long run

How Solow model is different from Chapter 3's model

1. ***K*** is no longer fixed: investment causes it to grow, depreciation causes it to shrink
2. ***L*** is no longer fixed: population growth causes it to grow
3. the consumption function is simpler

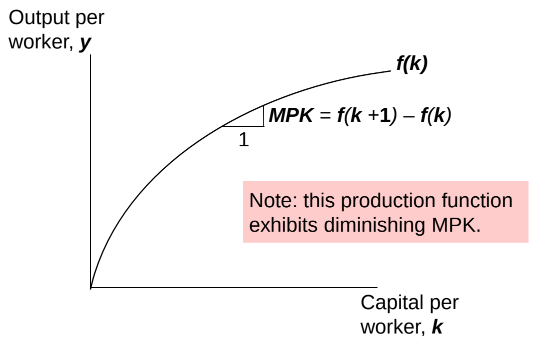
How Solow model is different from Chapter 3's model

4. no ***G*** or ***T*** (only to simplify presentation; we can still do fiscal policy experiments)
5. cosmetic differences

The production function

- In aggregate terms: $Y = F(K, L)$
- Define: $y = Y/L$ = output per worker
  $k = K/L$ = capital per worker
- Assume diminishing MPK
- Assume constant returns to scale:
  $zY = F(zK, zL)$ for any $z > 0$
- Pick $z = 1/L$. Then
  $Y/L = F(K/L, 1)$
  $y = F(k, 1)$
  $y = f(k)$ where $f(k) = F(k, 1)$

The production function



Note: this production function exhibits diminishing MPK.

The national income identity

- $Y = C + I$ (remember, no ***G***)
- In "per worker" terms:
  $y = c + i$
  where $c = C/L$ and $i = I/L$

The consumption function

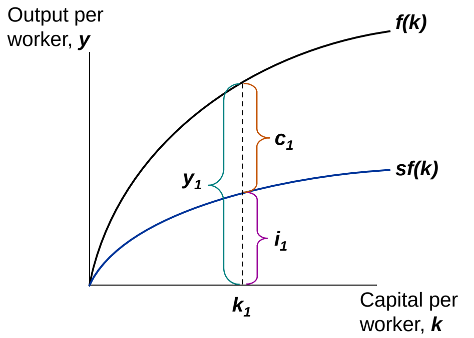
- s = the saving rate,
  the fraction of income that is saved
  (s is an exogenous parameter)

  Note: s is the *only* lowercase variable that is *not equal to* its uppercase version divided by L
- Consumption function: $c = (1–s)y$
  (per worker)

Saving and investment

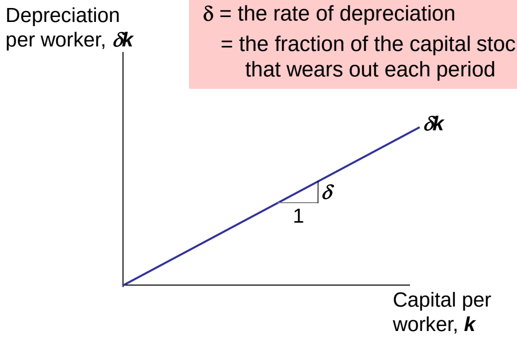
- saving (per worker) $= y – c$
  $= y – (1–s)y$
  $= sy$
- National income identity is $y = c + i$
  Rearrange to get: $i = y – c = sy$
  (investment = saving, like in chap. 3!)
- Using the results above,
  $i = sy = sf(k)$

Output, consumption, and investment



Depreciation

δ = the rate of depreciation
= the fraction of the capital stock that wears out each period



Capital accumulation

The basic idea: Investment increases the capital stock, depreciation reduces it.

Change in capital stock = investment – depreciation
$\Delta k = i – \delta k$

Since $i = sf(k)$, this becomes:

$$\boxed{\Delta k = s f(k) – \delta k}$$

The equation of motion for *k*

$$\boxed{\Delta k = s f(k) – \delta k}$$

- The Solow model's central equation
- Determines behavior of capital over time…
- …which, in turn, determines behavior of all of the other endogenous variables because they all depend on k. *E.g.*,
  income per person: $y = f(k)$
  consumption per person: $c = (1–s) f(k)$

The steady state

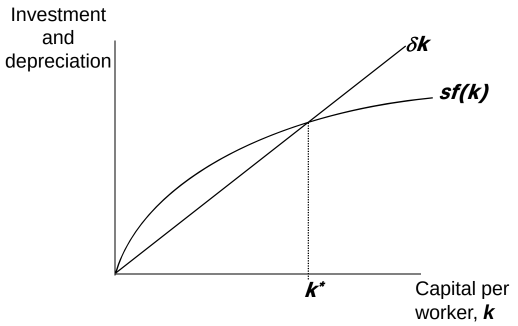
$$\boxed{\Delta k \ = \ s f(k) \ - \ \delta k}$$

If investment is just enough to cover depreciation [$sf(k) = \delta k$],
then capital per worker will remain constant:

$\Delta k = 0$.

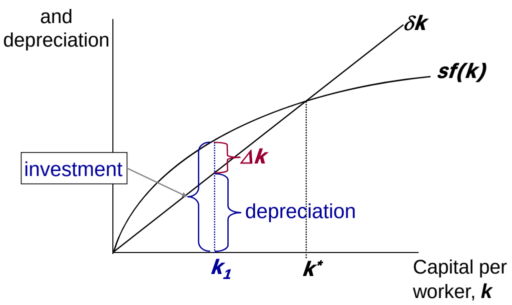
This occurs at one value of k, denoted k^*,
called the ***steady state capital stock***.

The steady state



Moving toward the steady state

$\Delta k = sf(k) - \delta k$



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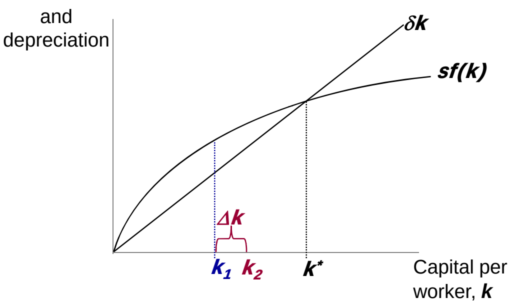

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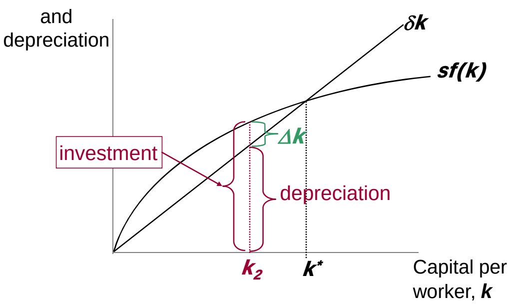

Moving toward the steady state

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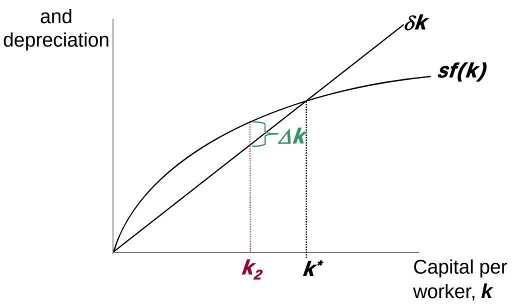
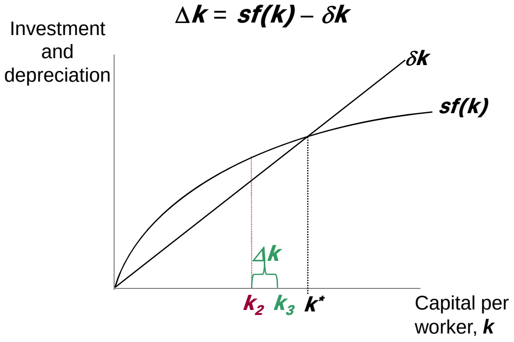


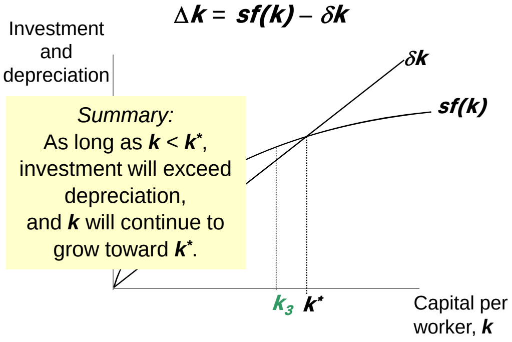
Moving toward the steady state

$$\Delta k = sf(k) - \delta k$$



Moving toward the steady state

$$\Delta k = sf(k) - \delta k$$



Summary:
As long as $k < k^*$,
investment will exceed
depreciation,
and k will continue to
grow toward k^*.

NOW YOU TRY:

Approaching k^* from above

Draw the Solow model diagram, labeling the steady state k^*.

On the horizontal axis, pick a value greater than k^* for the economy's initial capital stock. Label it k_1.

Show what happens to k over time. Does k move toward the steady state or away from it?

A numerical example

Production function (aggregate):

$$Y = F(K, L) = \sqrt{K \times L} = K^{1/2}L^{1/2}$$

To derive the per-worker production function, divide through by L:

$$\frac{Y}{L} = \frac{K^{1/2}L^{1/2}}{L} = \left(\frac{K}{L}\right)^{1/2}$$

Then substitute $y = Y/L$ and $k = K/L$ to get

$$y = f(k) = k^{1/2}$$

A numerical example, *cont.*

Assume:

- $s = 0.3$
- $\delta = 0.1$
- initial value of $k = 4.0$

Approaching the steady state: *A numerical example*

Assumptions: $y = \sqrt{k}$; $s = 0.3$; $\delta = 0.1$; initial $k = 4.0$

Year	k	y	c	i	δk	deltak
1	4.000	2.000	1.400	0.600	0.400	0.200
2	4.200	2.049	1.435	0.615	0.420	0.195
3	4.395	2.096	1.467	0.629	0.440	0.189
4	4.584	2.141	1.499	0.642	0.458	0.184
…						
10	5.602	2.367	1.657	0.710	0.560	0.150
…						
25	7.351	2.706	1.894	0.812	0.732	0.080
…						
100	8.962	2.994	2.096	0.898	0.896	0.002
…						
∞	9.000	3.000	2.100	0.900	0.900	0.000

NOW YOU TRY:

Solve for the Steady State

Continue to assume

$s = 0.3, \quad \delta = 0.1, \quad \text{and} \quad y = k^{1/2}$

Use the equation of motion

$$\Delta k = s\, f(k) - \delta k$$

to solve for the steady-state values of ***k***, ***y***, and ***c***.

ANSWERS:

Solve for the Steady State

$\Delta k = 0$ def. of steady state

$s\, f(k^*) = \delta k^*$ eq'n of motion with $\Delta k = 0$

$0.3\sqrt{k^*} = 0.1 k^*$ using assumed values

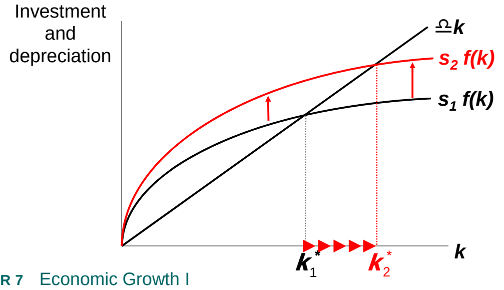
$$3 = \frac{k^*}{\sqrt{k^*}} = \sqrt{k^*}$$

Solve to get: $k^* = 9$ and $y^* = \sqrt{k^*} = 3$

Finally, $c^* = (1 - s)y^* = 0.7 \times 3 = 2.1$

An increase in the saving rate

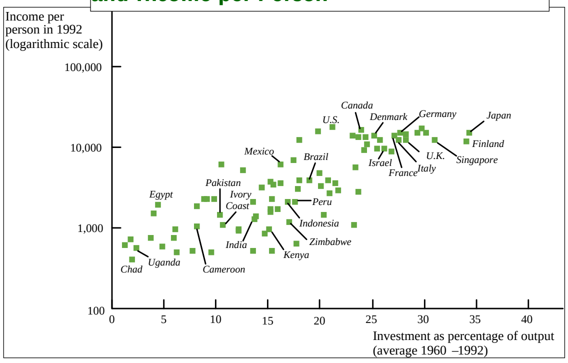
An increase in the saving rate raises investment…
…causing ***k*** to grow toward a new steady state:



Prediction:

- Higher ***s*** ⇒ higher k^*.
- And since $y = f(k)$,
  higher k^* ⇒ higher y^* .
- Thus, the Solow model predicts that countries with higher rates of saving and investment will have higher levels of capital and income per worker in the long run.

International Evidence on Investment Rates and Income per Person



The Golden Rule: Introduction

- Different values of ***s*** lead to different steady states. How do we know which is the “best” steady state?
- The “best” steady state has the highest possible consumption per person: $c^* = (1-s)\, f(k^*)$.
- An increase in ***s***
  - leads to higher k^* and y^*, which raises c^*
  - reduces consumption’s share of income (1–***s***), which lowers c^*.
- So, how do we find the ***s*** and k^* that maximize c^*?

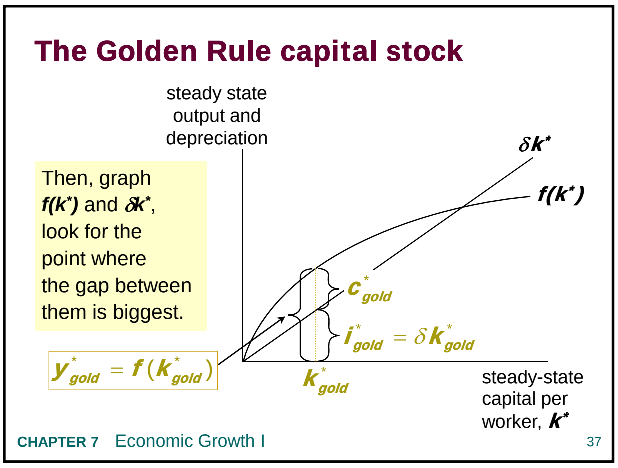
The Golden Rule capital stock

k^*_{gold} = the **Golden Rule level of capital**, the steady state value of k that maximizes consumption.

To find it, first express c^* in terms of k^*:

$$c^* = y^* - i^*$$
$$= f(k^*) - i^*$$
$$= f(k^*) - \delta k^*$$

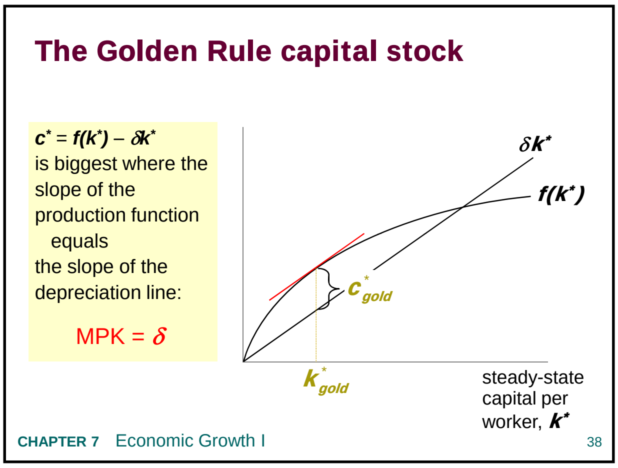
In the steady state: $i^* = \delta k^*$ because $\Delta k = 0$.



The Golden Rule capital stock

Then, graph $f(k^*)$ and δk^*, look for the point where the gap between them is biggest.

$y^*_{gold} = f(k^*_{gold})$



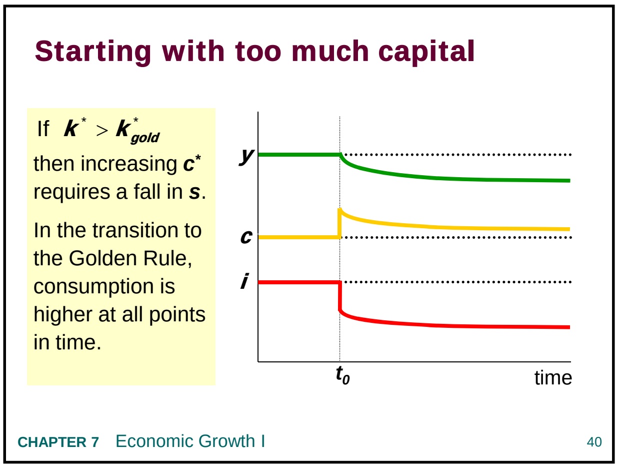
The Golden Rule capital stock

$c^* = f(k^*) - \delta k^*$ is biggest where the slope of the production function equals the slope of the depreciation line:

MPK = δ

The transition to the Golden Rule steady state

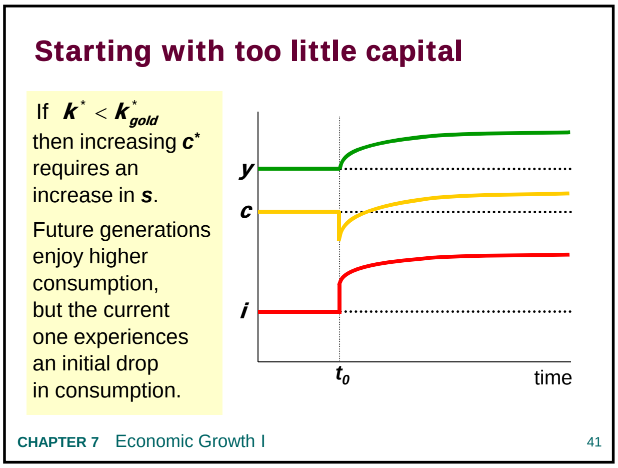
- The economy does NOT have a tendency to move toward the Golden Rule steady state.
- Achieving the Golden Rule requires that policymakers adjust s.
- This adjustment leads to a new steady state with higher consumption.
- But what happens to consumption during the transition to the Golden Rule?



Starting with too much capital

If $k^* > k^*_{gold}$ then increasing c^* requires a fall in s.

In the transition to the Golden Rule, consumption is higher at all points in time.



Starting with too little capital

If $k^* < k^*_{gold}$ then increasing c^* requires an increase in s.

Future generations enjoy higher consumption, but the current one experiences an initial drop in consumption.

Population growth

- Assume the population and labor force grow at rate ***n*** (exogenous):

$$\frac{\Delta L}{L} = n$$

- EX: Suppose ***L*** = 1,000 in year 1 and the population is growing at 2% per year (***n*** = 0.02).
- Then $\Delta L = nL = 0.02 \times 1{,}000 = 20$, so ***L*** = 1,020 in year 2.

Break-even investment

- $(\delta + n)k$ = **break-even investment**, the amount of investment necessary to keep ***k*** constant.
- Break-even investment includes:
  - δk to replace capital as it wears out
  - nk to equip new workers with capital

  (Otherwise, ***k*** would fall as the existing capital stock is spread more thinly over a larger population of workers.)

The equation of motion for *k*

- With population growth, the equation of motion for ***k*** is:

$$\Delta k = \underbrace{s f(k)}_{\text{actual investment}} - \underbrace{(\delta + n)\, k}_{\text{break-even investment}}$$

Why

- k(t+1)-k(t)=K(t+1)/L(t+1) –K(t)/L(t)

=1/L(t){K(t+1)/(1+n) -K(t)}

=(1/L(t))*(1/(1+n))*{K(t+1)-K(t)-nK(t)}

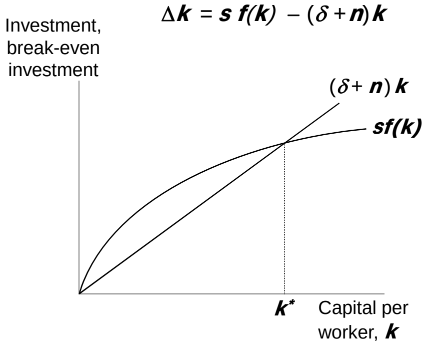
=(1/{(1+n)L(t)}) *{sY(t)- δK(t)-nK(t)}

=(1/(1+n))*{sy(t)-(δ + ***n***) ***k}***

This implies that k(t+1)-k(t) is zero if and only if s f(k) – (δ + n) k=0

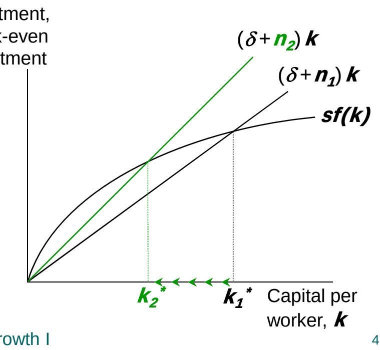
The Solow model diagram

$$\Delta k = s\, f(k) - (\delta + n)\, k$$



The impact of population growth

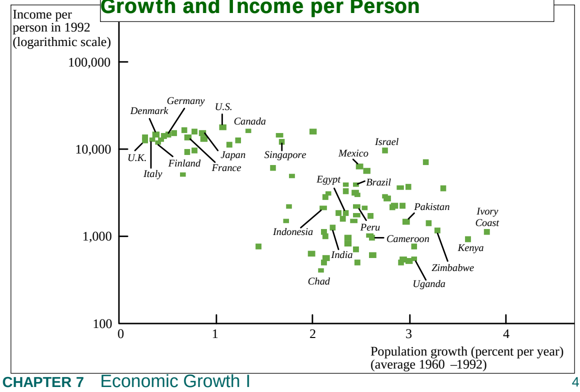
An increase in ***n*** causes an increase in break-even investment, leading to a lower steady-state level of ***k***.



Prediction:

- Higher ***n*** ⇒ lower ***k****.
- And since ***y*** = f(***k***) ,
  lower ***k**** ⇒ lower ***y****.
- Thus, the Solow model predicts that countries with higher population growth rates will have lower levels of capital and income per worker in the long run.

International Evidence on Population Growth and Income per Person



The Golden Rule with population growth

To find the Golden Rule capital stock, express c^* in terms of k^*:

$$c^* = y^* - i^*$$

$$= f(k^*) - (\delta + n)\, k^*$$

c^* is maximized when

$$\text{MPK} = \delta + n$$

or equivalently,

$$\boxed{\text{MPK} - \delta = n}$$

In the Golden Rule steady state, the marginal product of capital net of depreciation equals the population growth rate.

Alternative perspectives on population growth

The Malthusian Model (1798)

- Predicts population growth will outstrip the Earth's ability to produce food, leading to the impoverishment of humanity.
- Since Malthus, world population has increased sixfold, yet living standards are higher than ever.
- Malthus neglected the effects of technological progress.

Alternative perspectives on population growth

The Kremerian Model (1993)

- Posits that population growth contributes to economic growth.
- More people = more geniuses, scientists & engineers, so faster technological progress.
- Evidence, from very long historical periods:
  - As world pop. growth rate increased, so did rate of growth in living standards
  - Historically, regions with larger populations have enjoyed faster growth.